

A LIE GRADING WHICH IS NOT A SEMIGROUP GRADING

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Patera and Zassenhaus [PZ89] define a *Lie grading* as a decomposition of a Lie algebra into a direct sum of subspaces

$$\mathcal{L} = \oplus_{g \in G} \mathcal{L}_g, \quad (1)$$

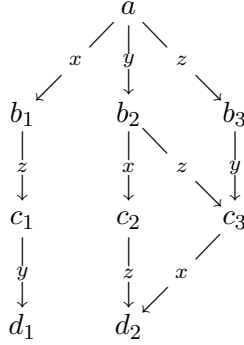
such that $\mathcal{L}_g \neq 0$ for any $g \in G$, and for any $g, g' \in G$, either $[\mathcal{L}_g, \mathcal{L}_{g'}] = 0$ or there exists a $g'' \in G$ such that $0 \neq [\mathcal{L}_g, \mathcal{L}_{g'}] \subseteq \mathcal{L}_{g''}$.

Then, in [PZ89, Theorem 1.(d)], it is asserted that, given a Lie grading (1), the set G embeds in an abelian semigroup so that the following property holds:

(P) For any $g, g', g'' \in G$ with $0 \neq [\mathcal{L}_g, \mathcal{L}_{g'}] \subseteq \mathcal{L}_{g''}$, $g + g' = g''$ holds in the semigroup.

The purpose of this note is to give a counterexample to this assertion. The problem in the proof of [PZ89, Theorem 1.(d)] lies in rule III of [PZ89, page 104].

Let V be a nine dimensional vector space over a field k with a fixed basis $\{a, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2\}$, and consider the endomorphisms x, y and z of V whose action on the basic elements is given in the following diagram



(Thus, for instance, $x(a) = b_1$, $x(b_2) = c_2$, $x(c_3) = d_2$ and x annihilates all the other basic elements.)

The associative subalgebra of $\text{End}_k(V)$ generated by these three endomorphisms is

$$A = \text{span} \{x, y, z, xy, xz, zx, yz, zy, yzx, xyz\}.$$

Note that $yx = 0 = x^2 = y^2 = z^2$, $xyz = xzy = zxy$, $xAx = yAy = zAz = 0$, and $A^4 = 0$. The elements in the spanning set of A given above are

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linearly independent, for if

$$\begin{aligned} \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 xy + \alpha_5 xz + \alpha_6 zx + \alpha_7 yz \\ + \alpha_8 zy + \alpha_9 yzx + \alpha_{10} xyz = 0, \end{aligned}$$

for some scalars $\alpha_i \in k$, this linear combination applied to a gives $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_6 = \alpha_9 = \alpha_{10} = 0$, as well as $\alpha_7 + \alpha_8 = 0$. Now applied to b_2 gives $\alpha_5 = 0$, and to b_1 $\alpha_7 = 0$. Therefore, the dimension of A is exactly 10.

Note that

$$[[x, y], z] = xyz - yxz - zxy + zyx = xyz - zxy = 0 \quad (\text{as } yx = 0),$$

$$\begin{aligned} [[y, z], x] &= yzx - zyx - xyz + xzy \\ &= yzx - 0 - (xyz - xzy) = yzx \quad (\text{as } xyz = xzy), \end{aligned}$$

$$[[z, x], y] = zxy - xzy - yzx + yxz = -yzx.$$

so the Lie subalgebra of $\text{End}_k(V)$ generated by x, y and z is

$$\mathfrak{g} = \text{span} \{x, y, z, [x, y] = xy, [x, z], [y, z], [[y, z], x]\},$$

which is a seven dimensional nilpotent Lie algebra.

Now, the Lie algebra

$$\mathcal{L} = \mathfrak{g} \oplus V \quad (\text{semidirect sum})$$

is a nilpotent Lie algebra, and its basis

$$B = \{x, y, z, [x, y], [x, z], [y, z], [[y, z], x], a, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2\}$$

satisfies that the bracket of any two elements in B is either 0 or a scalar multiple of another basic element. Hence, this basis gives a Lie grading:

$$\mathcal{L} = \bigoplus_{u \in B} \mathcal{L}_u, \quad (2)$$

where $\mathcal{L}_u = ku$ for any $u \in B$.

However, B is not contained in any grading abelian semigroup satisfying property (P) above, because

$$[y, [z, [x, a]]] = d_1, \quad \text{while} \quad [x, [y, [z, a]]] = d_2,$$

and d_1 and d_2 are in different homogeneous components in (2). If B were contained in an abelian semigroup satisfying (P), with addition denoted by \boxplus , then

$$d_1 = y \boxplus z \boxplus x \boxplus a = x \boxplus y \boxplus z \boxplus a = d_2$$

would hold, a contradiction.

REFERENCES

- [PZ89] J. Patera and H. Zassenhaus, *On Lie gradings. I*, Linear Algebra Appl. **112** (1989), 87–159.

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